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The Eulerian representations of Σ_n as restrictions of representations of Σ_{n+1}

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Abstract

We show that each Eulerian representation of Σ_n is the restriction of a representation of Σ_{n+1}. We describe the new representations, giving character formulae, and identify the one which restricts to the first Eulerian representation as the tree representation.

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0. Introduction

The Eulerian idempotents $e_n^{(j)}$, for $j = 1, \dots, n$, lying in the rational group algebra of the symmetric group $\mathbb{Q}\Sigma_n$, were defined by Gerstenhaber and Schack as follows [3]. An $(i, n-i)$-shuffle in Σ_n is a permutation π such that $\pi(1) < \pi(2) < \dots < \pi(i)$ and $\pi(i+1) < \pi(i+2) < \dots < \pi(n)$. Let $s_{i,n-i} = \sum (sgn\,\pi)\pi \in \mathbb{Q}\Sigma_n$, where the sum is over $(i, n-i)$-shuffles in Σ_n, and let $s_n = \sum_{i=1}^{n-1} s_{i,n-i} \in \mathbb{Q}\Sigma_n$. Now s_n has minimum polynomial $\prod_{j=1}^{n}(x - \mu_j)$, where $\mu_j = 2^j - 2$. Then define

$$e_n^{(j)} = \prod_{i \neq j} \frac{s_n - \mu_i}{(\mu_j - \mu_i)}.$$

The $e_n^{(j)}$'s for $j = 1, \dots, n$ form a family of mutually orthogonal idempotents such that $\sum_{j=1}^{n} e_n^{(j)} = 1$ [3, Theorem 1.2].

These idempotents provide decompositions of Hochschild and cyclic homology of a commutative algebra over a ground ring which contains $\mathbb{Q}$ [3, 8]. We briefly recall the definitions since in particular we will need a property of Connes' B map later.

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For A an associative algebra over k and M an A-bimodule, the Hochschild complex is $C_n(A;M) = M \otimes A^{\otimes n}$, with boundary $b : C_n(A;M) \to C_{n-1}(A;M)$ given by

$$\begin{aligned} b(m \otimes a_1 \otimes \cdots \otimes a_n) = {} & (ma_1 \otimes a_2 \otimes \cdots \otimes a_n) \\ & + \sum_{i=1}^{n-1} (-1)^i (m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ & + (-1)^n (a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

Here $\otimes$ denotes $\otimes_k$. The Hochschild homology of A with coefficients in M, denoted $HH_n(A;M)$, is the homology of this chain complex. The symmetric group Σ_n acts on the left on $C_n(A;M)$ by

$$\sigma(m \otimes a_1 \otimes \ldots \otimes a_n) = (m \otimes a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n)}),$$

and this extends linearly to an action of the group algebra $k\Sigma_n$. Then if A is commutative and the ground ring k contains $\mathbb{Q}$, the Eulerian idempotents commute with the Hochschild boundary map b, $be_n^{(j)} = e_{n-1}^{(j)} b$, so that they yield a decomposition of Hochschild homology. The first part of this decomposition, given by the idempotents $e_n^{(1)}$, is Harrison homology [3].

Letting $\bar{A} = A/k$, we may define the cyclic homology of A over k, denoted $HC_*(A)$, as the homology of the total complex corresponding to the normalised $(b - B)$ bicomplex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
b\downarrow & & b\downarrow & & b\downarrow & & \\
A \otimes \bar{A}^{\otimes n} & \xleftarrow{B} & A \otimes \bar{A}^{\otimes (n-1)} & \xleftarrow{B} & A \otimes \bar{A}^{\otimes (n-2)} & \xleftarrow{B} & \cdots \\
b\downarrow & & b\downarrow & & b\downarrow & & \\
\vdots & & \vdots & & \vdots & & \\
b\downarrow & & b\downarrow & & & & \\
A \otimes \bar{A} & \xleftarrow{\quad B \quad} & A & & & & \\
b\downarrow & & & & & & \\
A & & & & & &
\end{array}$$

where $B : A \otimes \bar{A}^{\otimes n} \to A \otimes \bar{A}^{(n+1)}$ is defined by

$$\begin{aligned} & B(a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1}) \\ & = \sum_{j=1}^{n+1} (-1)^{n(j-1)} (1 \otimes a_j \otimes a_{j+1} \otimes \ldots \otimes a_{n+1} \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_{j-1}). \end{aligned}$$

Now, for a commutative algebra A, over a ground ring k containing $\mathbb{Q}$, the Eulerian idempotents are well-behaved with respect to B as well as b, $Be_{n-1}^{(j-1)} = e_n^{(j)} B$, so that they decompose cyclic homology [8, 4.6.7].

The representation of Σ_n given by the right ideal $e_n^{(j)}\mathbb{Q}\Sigma_n$, which we denote by $E_n^{(j)}$, has been studied by Hanlon, who gives a character formula [5]. We show that this representation is the restriction of a representation of Σ_{n+1}, denoted $F_{n+1}^{(j)}$, given by a closely related idempotent $f_{n+1}^{(j)}$ in $\mathbb{Q}\Sigma_{n+1}$. By first finding a simplified formula for the product $e_n^{(j)}e_{n+1}^{(j)}$ in $\mathbb{Q}\Sigma_{n+1}$, we give a description of the representation $F_{n+1}^{(j)}$ as a virtual representation:

$$F_{n+1}^{(j)} \oplus \bigoplus_{i=1}^{j} E_{n+1}^{(i)} \cong \bigoplus_{i=1}^{j} Ind_{\Sigma_n}^{\Sigma_{n+1}} E_n^{(i)}.$$

This leads to a character formula using Hanlon's results. In the case $j = 1$, $F_{n+1}^{(1)}$ is the tree representation [11].

1. The idempotents $f_{n+1}^{(j)}$

We denote by λ_{n+1} the $n+1$ cycle $(1\ 2 \ldots n+1)$ in Σ_{n+1} and let $\Lambda_{n+1} = \frac{1}{n+1}\sum_{i=0}^{n}(sgn\,\lambda_{n+1}^i)\lambda_{n+1}^i \in \mathbb{Q}\Sigma_{n+1}$. We will always regard Σ_n as contained in Σ_{n+1} as the subgroup of permutations fixing $n+1$, and similarly $\mathbb{Q}\Sigma_n \subset \mathbb{Q}\Sigma_{n+1}$.

Proposition 1.1. $\Lambda_{n+1}s_n = s_n\Lambda_{n+1}$.

Proof. A typical term on the right-hand side of this equation is $\pi\lambda_{n+1}^j$, appearing with sign, $sgn(\pi).sgn(\lambda_{n+1}^j) = sgn(\pi\lambda_{n+1}^j)$, where π is some shuffle in Σ_n. Now we may write $\pi\lambda_{n+1}^j = \lambda_{n+1}^{\pi(j)}\pi'$, where $\pi' = \lambda_{n+1}^{-\pi(j)}\pi\lambda_{n+1}^j$ is in Σ_n. Let $\theta_j : \Sigma_n \to \Sigma_n$ be defined by $\theta_j(\pi) = \lambda_{n+1}^{-\pi(j)}\pi\lambda_{n+1}^j$. When $j = n+1$ we simply have the identity map, and for $j = n$ it was proved by Natsume and Schack that θ_n is a bijection which takes shuffles to shuffles [9, Lemma 9]. Since it is easily checked that $\theta_{n-k} = (\theta_n)^{k+1}$, the same holds for each θ_j. So each term of the right-hand side, $\pi\lambda_{n+1}^j$ with sign, appears in the left-hand side as $\lambda_{n+1}^{\pi(j)}\pi'$, with π' a shuffle, and with sign $sgn(\lambda_{n+1}^{\pi(j)}).sgn(\pi') = sgn(\lambda_{n+1}^{\pi(j)}\pi') = sgn(\pi\lambda_{n+1}^j)$. □

Corollary 1.2. *$\Lambda_{n+1}e_n^{(j)} = e_n^{(j)}\Lambda_{n+1}$ for $j = 1,\ldots,n$.*

Proof. Each $e_n^{(j)}$ is a polynomial in s_n, so this is immediate from the above. □

Thus, $\Lambda_{n+1}e_n^{(j)}$ is an idempotent in $\mathbb{Q}\Sigma_{n+1}$.

Definition 1.3. We denote by $f_{n+1}^{(j)}$ the idempotent element $\Lambda_{n+1}e_n^{(j)}$ in $\mathbb{Q}\Sigma_{n+1}$, for $j = 1,\ldots,n$. We let $E_n^{(j)}$ and $F_n^{(j)}$ denote the $\mathbb{Q}\Sigma_n$-modules $e_n^{(j)}\mathbb{Q}\Sigma_n$ and $f_n^{(j)}\mathbb{Q}\Sigma_n$, respectively.

Proposition 1.4. *The representation $F_{n+1}^{(j)}$ of Σ_{n+1} when restricted to a representation of Σ_n is isomorphic to $E_n^{(j)}$.*

Proof. Consider the homomorphism of right $\mathbb{Q}\Sigma_n$-modules $\theta : E_n^{(j)} \to F_{n+1}^{(j)}$ given by left multiplication by Λ_{n+1}. Now since Λ_{n+1} and $e_n^{(j)}$ commute, and since we may write $\pi \in \Sigma_{n+1}$ uniquely as $\lambda_{n+1}^i \pi'$ for some i and some $\pi' \in \Sigma_n$, we have $\Lambda_{n+1}(sgn\,\lambda_{n+1}^i)e_n^{(j)}\pi' = (sgn\,\lambda_{n+1}^i)f_{n+1}^{(j)}\pi' = f_{n+1}^{(j)}\lambda_{n+1}^i\pi' = f_{n+1}^{(j)}\pi$. Hence, the homomorphism of right $\mathbb{Q}\Sigma_n$-modules which is given by $f_{n+1}^{(j)}\pi \mapsto (sgn\,\lambda_{n+1}^i)e_n^{(j)}\pi'$ for $\pi \in \Sigma_{n+1}$ as above, is an inverse for θ. So $F_{n+1}^{(j)}$ and $E_n^{(j)}$ are isomorphic as $\mathbb{Q}\Sigma_n$-modules as required. □

Proposition 1.5.

$$\bigoplus_{j=1}^{n} F_{n+1}^{(j)} \cong Ind_{<\lambda_{n+1}>}^{\Sigma_{n+1}}(\varepsilon),$$

where ε denotes the sign representation of the cyclic subgroup $\langle\lambda_{n+1}\rangle$ of Σ_{n+1}.

Proof.

$$\sum_{j=1}^{n} f_{n+1}^{(j)} = \Lambda_{n+1}\sum_{j=1}^{n} e_n^{(j)} = \Lambda_{n+1},$$

since $\sum_{j=1}^{n} e_n^{(j)} = 1$ [3, Theorem 1.2]. So the sum of the representations $F_{n+1}^{(j)}$ is $\Lambda_{n+1}\mathbb{Q}\Sigma_{n+1}$. Since Λ_{n+1} is the standard idempotent for the sign representation of the cyclic subgroup of Σ_{n+1} generated by λ_{n+1}, $\Lambda_{n+1}\mathbb{Q}\Sigma_{n+1}$ is the claimed induced representation. □

2. A relation between $e_n^{(j)}$ and $e_{n+1}^{(j)}$

In this section we prove certain relations among the $e_n^{(j)}$'s and $f_n^{(j)}$'s, which will be needed in the following section to give descriptions of our representations. The main result is Proposition 2.5, giving a simplification of the product $e_n^{(j)}e_{n+1}^{(j)}$. We adopt the convention that $f_n^{(k)} = e_n^{(k)} = 0$ whenever $k \leq 0$ or $k > n$.

Lemma 2.1. $f_{n+1}^{(j-1)} = e_{n+1}^{(j)}\Lambda_{n+1}$, *for* $j = 1, \ldots, n+1$.

Proof. We need to show that $\Lambda_{n+1}e_n^{(j-1)} = e_{n+1}^{(j)}\Lambda_{n+1}$. We start from Loday's relation $Be_n^{(j-1)} = e_{n+1}^{(j)}B$ [8, Theorem 4.6.6] where B is Connes' boundary operator in the normalised setting. We may write B as $(n+1)s\Lambda_{n+1}$, where s is the standard extra degeneracy operator, $s(a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1}) = (1 \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1})$ [8, 1.1.12]. So we have $(n+1)s\Lambda_{n+1}e_n^{(j-1)} = (n+1)e_{n+1}^{(j)}s\Lambda_{n+1}$. Since the action of $e_{n+1}^{(j)}$ here is on the last $n+1$ places, leaving the first unaffected, $e_{n+1}^{(j)}s\Lambda_{n+1} = se_{n+1}^{(j)}\Lambda_{n+1}$. So $\Lambda_{n+1}e_n^{(j-1)} = e_{n+1}^{(j)}\Lambda_{n+1}$ as required. □

Corollary 2.2. $F_{n+1}^{(j-1)} \subset E_{n+1}^{(j)}$, *for* $j = 1, \ldots, n+1$.

Note that, since $F_{n+1}^{(j-1)}$ restricts to $E_n^{(j-1)}$, to understand the $E_n^{(j)}$'s it is sufficient to understand these submodules.

Since the group algebra $\mathbb{Q}\Sigma_{n+1}$ is semi-simple, we may write

$$E_{n+1}^{(j)} = F_{n+1}^{(j-1)} \oplus x\mathbb{Q}\Sigma_{n+1}$$

for some $x \in \mathbb{Q}\Sigma_{n+1}$.

Notation. Let $p_n = (1\ n)(2\ n-1)(3\ n-2)\ldots \in \Sigma_n$ and let $op_n = (-1)^{n(n+1)/2}p_n = (-1)^n(sgn\,p_n)p_n \in \mathbb{Q}\Sigma_n$. Now consider the idempotents $\sigma_n^{(j)} = \frac{1}{2}(1+(-1)^j op_n)$ in $\mathbb{Q}\Sigma_{\acute{n}}$. These two idempotents correspond to the sums of the even and odd Eulerian idempotents:

$$\sigma_n^{(j)} = \sum_{i \equiv j \pmod 2} e_n^{(i)}$$

by [3]. In particular, $\sigma_n^{(j)}$ (and, hence, p_n) is a polynomial in s_n. Of course, since the $e_n^{(i)}$'s are mutually orthogonal,

$$e_n^{(i)}\sigma_n^{(j)} = \sigma_n^{(j)}e_n^{(i)} = \begin{cases} e_n^{(i)} & \text{if } i \equiv j \ (mod\ 2), \\ 0 & \text{otherwise.} \end{cases}$$

Let $s_{n+1}^* = \sum(sgn\,\pi)\pi$, where the sum is over shuffles in Σ_{n+1} which do not fix $n+1$. So we may write $s_{n+1} = 1 + s_n + s_{n+1}^*$.

Lemma 2.3. $(1+s_{n+1}^*)\sigma_{n+1}^{(j)} = \sigma_n^{(j+1)}(1+s_{n+1}^*)$.

Proof. Equivalently, we show $p_n(1+s_{n+1}^*)p_{n+1} = (-1)^n(1+s_{n+1}^*)$. Firstly, on the left-hand side we have $p_n p_{n+1} = \lambda_{n+1}^{-1}$, and since this is a 1-shuffle with $\lambda_{n+1}^{-1}(1) = n+1$, it also appears in the right-hand side. Any other term on the left-hand side has the form $(sgn\,\pi)p_n\pi p_{n+1}$ where π is some k-shuffle in Σ_{n+1} not fixing $n+1$. Then $\pi(k) = n+1$, and it is easy to see that $p_n\pi p_{n+1}$ is an $(n+2-k)$-shuffle taking $(n+2-k)$ to $n+1$. (In the case $k = 1$, we must have $\pi = \lambda_{n+1}^{-1}$, and we get $p_n\pi p_{n+1} = 1$.) Since $sgn(p_n\pi p_{n+1}) = (-1)^n sgn(\pi)$, the result follows. □

Lemma 2.4. $e_n^{(j)}\sigma_{n+1}^{(j)}(s_{n+1}-\mu_j) = 0$.

Proof.

$$\begin{aligned} e_n^{(j)}\sigma_{n+1}^{(j)}(s_{n+1}-\mu_j) &= e_n^{(j)}(s_{n+1}-\mu_j)\sigma_{n+1}^{(j)} \\ &= e_n^{(j)}(1+s_n+s_{n+1}^*-\mu_j)\sigma_{n+1}^{(j)} \\ &= e_n^{(j)}(s_n-\mu_j)\sigma_{n+1}^{(j)} + e_n^{(j)}(1+s_{n+1}^*)\sigma_{n+1}^{(j)} \\ &= e_n^{(j)}(1+s_{n+1}^*)\sigma_{n+1}^{(j)} \\ &= e_n^{(j)}\sigma_n^{(j+1)}(1+s_{n+1}^*) \quad \text{by Lemma 2.3} \\ &= 0. \quad \square \end{aligned}$$

Proposition 2.5. $e_n^{(j)}e_{n+1}^{(j)} = e_n^{(j)}\sigma_{n+1}^{(j)}$.

Proof. Recall that the minimum polynomial of s_{n+1} is $\prod_{j=1}^{n+1}(x-\mu_j)$, where $\mu_j = 2^j - 2$. So right multiplication by s_{n+1} is an operator on the subspace of $\mathbb{Q}\Sigma_{n+1}$ spanned by $1, s_{n+1}, \ldots, s_{n+1}^n$. It has $n+1$ distinct eigenvalues $\mu_1, \ldots, \mu_{n+1}$ and it follows from the definition of the Eulerian idempotents that right multiplication by $e_{n+1}^{(j)}$ is projection onto the eigenspace corresponding to eigenvalue μ_j. So, by Lemma 2.4, $e_n^{(j)}\sigma_{n+1}^{(j)}$ is contained in the left ideal $\mathbb{Q}\Sigma_{n+1}(e_{n+1}^{(j)})$. Hence, $e_n^{(j)}\sigma_{n+1}^{(j)} = e_n^{(j)}\sigma_{n+1}^{(j)}e_{n+1}^{(j)}$. But $\sigma_{n+1}^{(j)}e_{n+1}^{(j)} = e_{n+1}^{(j)}$. □

In fact, using the same method, one can also show that $e_n^{(j)}e_{n+1}^{(j)} = \sigma_n^{(j)}e_{n+1}^{(j)}$.

3. $F_{n+1}^{(j)}$ as a virtual representation

The main result of this section is Theorem 3.4, giving a description of the representation $F_{n+1}^{(j)}$.

Definition 3.1. We define certain elements of the group algebra $\mathbb{Q}\Sigma_{n+1}$:

$$x_{n+1}^{(j)} = \frac{2}{n+1}\left((n-1) + (-1)^j op_n + \sum_{i=3}^{n}(i-2)(-1)^{ni}\lambda_{n+1}^i\right).$$

Lemma 3.2.

$$(1+(-1)^{n+1}\lambda_{n+1}^{-1})x_{n+1}^{(j)}$$

$$= \frac{2}{n+1}\left((n-1) + (-1)^j op_n + (-1)^{n+j-1}\lambda_{n+1}^{-1}op_n - \sum_{i=2}^{n}(-1)^{ni}\lambda_{n+1}^i\right).$$

Proposition 3.3. $e_n^{(j)}e_{n+1}^{(j)}x_{n+1}^{(j)} = (1-\Lambda_{n+1})e_n^{(j)}$.

Proof. We will use $e_n^{(j)} = (-1)^j e_n^{(j)}op_n$ and $op_n op_{n+1} = (-1)^{n+1}\lambda_{n+1}^{-1}$. Now

$$\begin{aligned} e_n^{(j)}e_{n+1}^{(j)}x_{n+1}^{(j)} &= e_n^{(j)}\sigma_{n+1}^{(j)}x_{n+1}^{(j)} \quad \text{by Proposition 2.5} \\ &= \frac{1}{2}e_n^{(j)}(1+(-1)^j op_{n+1})x_{n+1}^{(j)} \\ &= \frac{1}{2}e_n^{(j)}(1+op_n op_{n+1})x_{n+1}^{(j)} \\ &= \frac{1}{2}e_n^{(j)}(1+(-1)^{n+1}\lambda_{n+1}^{-1})x_{n+1}^{(j)} \end{aligned}$$

So

$$\begin{aligned}
e_n^{(j)}e_{n+1}^{(j)}x_{n+1}^{(j)} &= \frac{e_n^{(j)}}{n+1}\Big((n-1)+(-1)^j op_n+(-1)^{n+j-1}\lambda_{n+1}^{-1}op_n \\
&\qquad -\sum_{i=2}^{n}(-1)^{ni}\lambda_{n+1}^{i}\Big) \\
&= \frac{e_n^{(j)}}{n+1}\Big(n+(-1)^{n+1}op_n\lambda_{n+1}^{-1}op_n-\sum_{i=2}^{n}(-1)^{ni}\lambda_{n+1}^{i}\Big) \\
&= \frac{e_n^{(j)}}{n+1}\Big(n+(-1)^{n+1}\lambda_{n+1}-\sum_{i=2}^{n}(-1)^{ni}\lambda_{n+1}^{i}\Big) \\
&= \frac{e_n^{(j)}}{n+1}\Big(n-\sum_{i=1}^{n}(-1)^{ni}\lambda_{n+1}^{i}\Big) \\
&= e_n^{(j)}(1-\Lambda_{n+1}) \\
&= (1-\Lambda_{n+1})e_n^{(j)} \quad \text{by Corollary 1.2.} \quad \Box
\end{aligned}$$

Now we can prove the main result. We will use the fact that given an idempotent e in $\mathbb{Q}\Sigma_n$, giving a representation $e\mathbb{Q}\Sigma_n$ of Σ_n, then the induced representation of Σ_{n+1} is given by $e\mathbb{Q}\Sigma_{n+1}$. We also need the result, due to Hanlon, that the dimension of the representation $E_n^{(j)}$ is $s(n,j)$, the number of permutations in Σ_n with exactly j cycles [5].

Theorem 3.4.

$$F_{n+1}^{(j)} \oplus \bigoplus_{i=1}^{j} E_{n+1}^{(i)} \cong \bigoplus_{i=1}^{j} Ind_{\Sigma_n}^{\Sigma_{n+1}} E_n^{(i)}.$$

Proof. The result will be proved by induction on j. First, we consider the case $j=1$. Here we need to show that $F_{n+1}^{(1)} \oplus E_{n+1}^{(1)} \cong Ind_{\Sigma_n}^{\Sigma_{n+1}} E_n^{(1)}$. That is,

$$F_{n+1}^{(1)} \oplus E_{n+1}^{(1)} \cong e_n^{(1)}\mathbb{Q}\Sigma_{n+1}.$$

Now it is clear that

$$\begin{aligned}
e_n^{(1)}\mathbb{Q}\Sigma_{n+1} &= \Lambda_{n+1}e_n^{(1)}\mathbb{Q}\Sigma_{n+1} \oplus (1-\Lambda_{n+1})e_n^{(1)}\mathbb{Q}\Sigma_{n+1} \\
&= F_{n+1}^{(1)} \oplus (1-\Lambda_{n+1})e_n^{(1)}\mathbb{Q}\Sigma_{n+1}.
\end{aligned}$$

So we must show that $(1-\Lambda_{n+1})e_n^{(1)}\mathbb{Q}\Sigma_{n+1} \cong E_{n+1}^{(1)}$, that is $(1-\Lambda_{n+1})e_n^{(1)}\mathbb{Q}\Sigma_{n+1} \cong e_{n+1}^{(1)}\mathbb{Q}\Sigma_{n+1}$. Using the fact that $E_n^{(1)}$ has dimension $(n-1)!$, we see that both of these modules have dimension $n!$. We define

$$\theta : e_{n+1}^{(1)}\mathbb{Q}\Sigma_{n+1} \to (1-\Lambda_{n+1})e_n^{(1)}\mathbb{Q}\Sigma_{n+1}$$

to be the homomorphism of right $\mathbb{Q}\Sigma_{n+1}$-modules given by left-multiplication by the element $(1 - \Lambda_{n+1})e_n^{(1)}$. Then

$$(1 - \Lambda_{n+1})e_n^{(1)} = e_n^{(1)}e_{n+1}^{(1)}x_{n+1}^{(1)} = (1 - \Lambda_{n+1})e_n^{(1)}e_{n+1}^{(1)}x_{n+1}^{(1)} = \theta(e_{n+1}^{(1)}x_{n+1}^{(1)}),$$

by Proposition 3.3. Thus, θ is surjective, and so an isomorphism, giving the result for j=1. Note that this identifies $E_{n+1}^{(1)}$ with the submodule $e_n^{(1)}(1 - \Lambda_{n+1})\mathbb{Q}\Sigma_{n+1}$ of $e_n^{(1)}\mathbb{Q}\Sigma_{n+1}$.

Now we assume the result for $j - 1$ and consider j. Using the induction hypothesis it is sufficient to show that $F_{n+1}^{(j)} \oplus E_{n+1}^{(j)} \cong F_{n+1}^{(j-1)} \oplus Ind_{\Sigma_n}^{\Sigma_{n+1}} E_n^{(j)}$. That is,

$$\Lambda_{n+1}e_n^{(j)}\mathbb{Q}\Sigma_{n+1} \oplus E_{n+1}^{(j)} \cong F_{n+1}^{(j-1)} \oplus e_n^{(j)}\mathbb{Q}\Sigma_{n+1}.$$

Now, we clearly have

$$e_n^{(j)}\mathbb{Q}\Sigma_{n+1} = \Lambda_{n+1}e_n^{(j)}\mathbb{Q}\Sigma_{n+1} \oplus (1 - \Lambda_{n+1})e_n^{(j)}\mathbb{Q}\Sigma_{n+1},$$

so we must show that $E_{n+1}^{(j)} \cong F_{n+1}^{(j-1)} \oplus (1 - \Lambda_{n+1})e_n^{(j)}\mathbb{Q}\Sigma_{n+1}$. By Corollary 2.2, $E_{n+1}^{(j)} = F_{n+1}^{(j-1)} \oplus x\mathbb{Q}\Sigma_{n+1}$. Hence, the above simplifies to showing that $x\mathbb{Q}\Sigma_{n+1} \cong (1 - \Lambda_{n+1})e_n^{(j)}\mathbb{Q}\Sigma_{n+1}$. We define $\theta : E_{n+1}^{(j)} \to (1 - \Lambda_{n+1})e_n^{(j)}\mathbb{Q}\Sigma_{n+1}$ to be the homomorphism of right $\mathbb{Q}\Sigma_{n+1}$-modules given by left-multiplication by $(1 - \Lambda_{n+1})e_n^{(j)}$. By Corollary 1.2, $(1 - \Lambda_{n+1})e_n^{(j)}\Lambda_{n+1}e_n^{(j-1)} = 0$, so $F_{n+1}^{(j-1)} \subset Ker\,\theta$. Hence, θ induces a $\mathbb{Q}\Sigma_{n+1}$-module homomorphism: $\theta' : x\mathbb{Q}\Sigma_{n+1} \to (1 - \Lambda_{n+1})e_n^{(j)}\mathbb{Q}\Sigma_{n+1}$. Next, we check the dimensions of these $\mathbb{Q}\Sigma_{n+1}$-modules. We have seen in Proposition 1.4 that $F_{n+1}^{(j)}$ restricts to $E_n^{(j)}$, so has the same dimension, $s(n,j)$. So $x\mathbb{Q}\Sigma_{n+1}$ has dimension $s(n+1,j) - s(n,j-1)$, and $(1 - \Lambda_{n+1})e_n^{(j)}\mathbb{Q}\Sigma_{n+1}$ has dimension $(n+1)s(n,j) - s(n,j) = ns(n,j)$. Since $s(n+1,j) = s(n,j-1) + ns(n,j)$, (see [4, p. 261, Eq. (6.8)], the two modules do have the same dimension. Hence, it is sufficient to show that θ' is surjective to conclude that it is a $\mathbb{Q}\Sigma_{n+1}$-module isomorphism. But,

$$\begin{aligned}(1 - \Lambda_{n+1})e_n^{(j)} &= e_n^{(j)}e_{n+1}^{(j)}x_{n+1}^{(j)} && \text{by Proposition 3.3}\\ &= (1 - \Lambda_{n+1})e_n^{(j)}e_{n+1}^{(j)}x_{n+1}^{(j)} && \text{since } 1 - \Lambda_{n+1} \text{ is an idempotent}\\ &= \theta'(e_{n+1}^{(j)}x_{n+1}^{(j)}).\end{aligned}$$

Hence, θ' is surjective. □

Notation. Let Ψ_{n+1}^j denote the character of the representation $F_{n+1}^{(j)}$ of Σ_{n+1} and let χ_n^j denote the character of the representation $E_n^{(j)}$ of Σ_n.

Corollary 3.5.

$$\Psi_{n+1}^j = \sum_{i=1}^{j} Ind_{\Sigma_n}^{\Sigma_{n+1}}(\chi_n^i) - \sum_{i=1}^{j} \chi_{n+1}^i.$$

We give the formula for the character Ψ_{n+1}^1 explicitly.

Corollary 3.6. *For $g \in \Sigma_{n+1}$,*

$$\Psi^1_{n+1}(g) = \begin{cases} (-1)^{n+s}(s-1)!(r)^{s-1}\mu(r) & \text{if } g \text{ has cycle type } (r)^s \text{ with } \\ & r > 1, \text{ or } (r)^s(1), \\ 0 & \text{if } g \text{ has any other cycle type.} \end{cases}$$

Proof. We have shown that $\Psi^1_{n+1} = Ind_{\Sigma_n}^{\Sigma_{n+1}}(\chi^1_n) - \chi^1_{n+1}$. The result is a straightforward induced character calculation from Hanlon's result: $\chi^1_n = \varepsilon \cdot \left(Ind_{C_n}^{\Sigma_n}\rho_n\right)$, where ρ_n is a faithful linear character of the cyclic subgroup of Σ_{n+1} generated by an n-cycle and ε is the alternating character [5]. □

It is shown in [11] that this is exactly the character of the tree representation of Σ_{n+1}. This representation arises in Γ-homology, $H\Gamma_*$, a homology theory for E_∞-ring spectra, introduced by Robinson in [10], which specialises to a new homology theory for commutative algebras. Let B be a flat commutative algebra over a commutative ground ring A and M a B-module. An application of our results allows us to prove that when the ground ring A contains $\mathbb{Q}$, Γ-homology agrees with Harrison homology, $H\Gamma_p(B/A;M) \cong Harr_{p+1}(B/A;M)$. (In general, the theories are different.) Since the definition of Γ-homology is rather long, here we only outline the idea of the proof. The result was announced in [14] and will appear elsewhere.

There is a first quadrant spectral sequence converging to Γ-homology,

$$\varepsilon^1_{p,q} = M \otimes \mathrm{Tor}_q^{A\Sigma_{p+1}}(V_{p+1}, B^{\otimes p+1}) \underset{p}{\Longrightarrow} H\Gamma_{p+q}(B/A;M),$$

where V_{p+1} denotes the restriction to Σ_{p+1} of the tree representation of Σ_{p+2}. When A contains $\mathbb{Q}$ the spectral sequence collapses to the edge and, by Proportion 1.4, V_{p+1} is isomorphic to $E^{(1)}_{p+1}$. Then Γ-homology is the homology of a complex

$$M \otimes E^{(1)}_{p+1} \otimes_{\Sigma_{p+1}} B^{\otimes p+1}$$

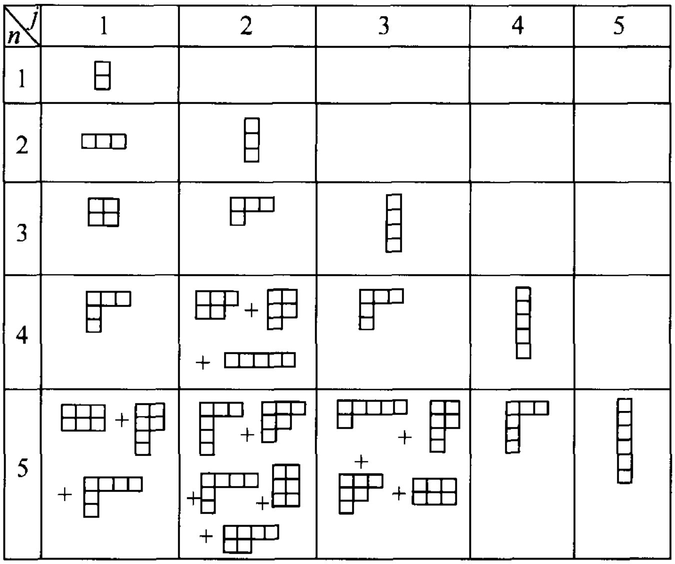
with a differential which can be identified as induced by the Hochschild boundary b. This gives the first part of the decomposition of Hochschild homology, namely Harrison homology.

The representation $F^{(1)}_{n+1}$ also occurs in the homology of partition lattices [12], the homology of configuration spaces [2, 7], and (up to sign) as the multilinear part of the free Lie algebra [1, 11].

4. Some results on decompositions

Table 1 lists the decompositions of the representations $F^{(j)}_{n+1}$ of Σ_{n+1} for $n = 1,\dots,5$ and $j = 1,\dots,n$. The first column gives $F^{(1)}_{n+1}$, the tree representation of Σ_{n+1}. The sum along the nth row of the table gives the representation $\Lambda_{n+1}\mathbb{Q}\Sigma_{n+1}$, the sign representation of $\langle\lambda_{n+1}\rangle$ induced to Σ_{n+1}.

Table 1
Decompositions of $F_{n+1}^{(j)}$



For some values of j, it is possible to describe the decomposition of the representation $F_{n+1}^{(j)}$ into irreducible components. Let ω^λ be the irreducible character of the symmetric group Σ_{n+1} corresponding to a partition λ of $n+1$. Let $a(T)$ denote the sum of ascents of a standard tableau T, that is the sum of those i such that $i+1$ appears to the right of i in T. Then we denote by $\lambda(j,n)$, the number of standard tableaux T of shape λ such that $a(T) \equiv j \pmod{n}$.

Proposition 4.1.

(1) $(j=1)$ *The multiplicity of* ω^λ *in* Ψ_{n+1}^1 *is* $\lambda(1,n) - \lambda(1,n+1)$,

(2) $(j=n-2)$ $\Psi_{n+1}^{n-2} = \omega^{2^2 1^{n-3}} \oplus \omega^{321^{n-4}} \oplus \omega^{3^2 1^{n-5}} \oplus \omega^{51^{n-4}}$,

(3) $(j=n-1)$ $\Psi_{n+1}^{n-1} = \omega^{31^{n-2}}$,

(4) $(j=n)$ $\Psi_{n+1}^{n} = \omega^{1^{n+1}}$, *and*

(5) *the multiplicity of* ω^λ *in the sum of characters* $\sum_{j=1}^{n} \Psi_{n+1}^{j}$ *is* $\lambda(0,n+1)$.

Proof. (1) It is easily seen that T is a standard tableau for Σ_{n+1} such that $a(T) \equiv 1 \pmod{n}$ if and only if it is obtained from a standard tableau T' for Σ_n, with $a(T') \equiv 1 \pmod{n}$, by attaching $n+1$ to the end of some row or column. Now in χ_n^1, ω^λ has multiplicity the number of standard tableaux T' of shape λ for Σ_n such that $a(T') \equiv 1 \pmod{n}$ by a result of Kraskiewicz and Weyman [6] So in $Ind_{\Sigma_n}^{\Sigma_{n+1}}(\chi_n^1)$, ω^λ has multiplicity $\lambda(1,n)$. Since, $\Psi_{n+1}^1 = Ind_{\Sigma_n}^{\Sigma_{n+1}}(\chi_n^1) - \chi_{n+1}^1$, the result follows.

(2–3–4) These results follow directly from those of [5] for $\chi_n^{n-2}, \chi_n^{n-1}$ and χ_n^n. That is, for $j = n-2, n-1, n$, the decomposition of Ψ_{n+1}^j given above is the only one which will restrict back to give the correct decomposition of χ_n^i. (Of course, in the case $j = n$, we have $f_{n+1}^{(n)} = \Lambda_{n+1}e_n^{(n)} = \Lambda_{n+1}\varepsilon_n = \varepsilon_{n+1}$, where $\varepsilon_n = \frac{1}{n!}\sum_{\pi\in\Sigma_n}(sgn\,\pi)\pi$, and we see directly that we have the sign representation.)

(5) We have seen that this sum of characters is just the sign character of $\langle\lambda_{n+1}\rangle$ induced to Σ_{n+1}. The formula for the decomposition can be deduced from the work of Stembridge [13]. □

We also give the relationship between our characters and the trivial character.

Proposition 4.2. *The trivial character ω^{n+1} appears only in $\Psi_{n+1}^{n/2}$ if n is even and does not appear in any Ψ_{n+1}^i if n is odd.*

Proof. Let $e_{n+1} = \frac{1}{(n+1)!}\sum_{\pi\in\Sigma_{n+1}}\pi$. It is easily checked that

$$\Lambda_{n+1}e_{n+1} = \begin{cases} e_{n+1} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Hence, the trivial representation does not appear in $\Lambda_{n+1}\mathbb{Q}\Sigma_{n+1}$ when n is odd. When n is even it appears once, and this must be in $\Lambda_{n+1}e_n^{(n/2)}\mathbb{Q}\Sigma_{n+1}$, since Hanlon shows that the trivial representation of Σ_n always appears in $e_n^{([(n+1)/2])}\mathbb{Q}\Sigma_n$. □

Corollary 4.3. *The character ω^{n1} does not appear in any Ψ_{n+1}^i if n is even and appears only in $\Psi_{n+1}^{(n+1)/2}$ if n is odd.*

Proof. The irreducible character ω^{n1} of Σ_{n+1} is the only one apart from ω^{n+1} which gives a copy of the trivial character of Σ_n on restriction. Hence, the result follows from the above. □

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